Comment on "Scaling behavior of classical wave transport in mesoscopic media at the localization transition"

N. Cherroret, S. E. Skipetrov, and B. A. van Tiggelen
Laboratoire de Physique et Modélisation des Milieux Condensés, Université Joseph Fourier, CNRS UMR 5493, BP 166, 25 rue des Martyrs, Maison des Magistères, 38042 Grenoble Cedex 09, France
(Received 23 September 2008; published 23 July 2009)

We emphasize the importance of the position dependence of the diffusion coefficient $D(\mathbf{r})$ in the self-consistent theory of localization and argue that the scaling law $T \propto \ln L/L^2$ obtained by Cheung and Zhang [Phys. Rev. B **72**, 235102 (2005)] for the average transmission coefficient T of a disordered slab of thickness L at the localization transition is an artifact of replacing $D(\mathbf{r})$ by its harmonic mean. The correct scaling $T \propto 1/L^2$ is obtained by properly treating the position dependence of $D(\mathbf{r})$.

DOI: 10.1103/PhysRevB.80.037101 PACS number(s): 42.25.Dd, 42.25.Bs, 72.15.Rn, 72.20.Ee

In a recent paper[1] Cheung and Zhang (CZ) apply the self-consistent (SC) theory of localization to study the transmission of waves through a slab of disordered medium at the Anderson localization transition. The SC theory is a powerful tool to deal with the phenomenon of Anderson localization, but its application to disordered media of finite size requires some care. In the original papers by Vollhardt and Wölfle,[2] the size L of disordered sample was acknowledged using a lower cutoff in the integration over momentum. Despite the obvious crudeness of this approach, it was sufficient to recover the main results of the scaling theory of localization[3] and added a great physical insight into the phenomenon of disorder-induced localization. Later on, van Tiggelen *et al.*[4] argued that in a medium of finite size the SC theory naturally leads to a position dependence of the diffusion coefficient $D(\mathbf{r})$. This adapted SC theory was successfully applied to study coherent backscattering[4] and dynamics[5,6] of localized waves. Microscopic justifications for position dependence of D have been recently presented based on the diagrammatic[7] and field-theoretic[8] calculations.

CZ propose a way of overcoming technical difficulties caused by the position dependence of $D(\mathbf{r})$ (Ref. 1) (see also Ref. 9). They average the equation for $1/D(\mathbf{r})$, their Eq. (1), over the sample volume, thus replacing $D(\mathbf{r})$ by its harmonic mean $\bar{D}$. In this Comment we argue that although such an approach can be justified in the weak localization regime,[9] it is not adequate at the mobility edge and in the Anderson localization regime. In particular, our calculations that properly treat the position dependence of $D(\mathbf{r})$, do not confirm the scaling law $T \propto \ln L/L^2$ found by CZ for the transmission coefficient T of a disordered slab of thickness L at the mobility edge. Instead, we find $T \propto 1/L^2$ in agreement with the scaling theory of localization.[3]

To study the scaling of the average transmission coefficient T with the thickness L of disordered slab, we solve the two equations of SC theory—Eqs. (1) and (2) of Ref. 6 with $\Omega=0$ (stationary regime) and $k\ell=1$ (mobility edge)[10]—numerically. We use the same boundary conditions and the same method of numerical solution as in Ref. 6 and vary the thickness of the slab L from $10^2\ell$ to $8\times10^3\ell$. Here k is the wave number of the wave and ℓ is the mean free path due to disorder. Our results are presented in Fig. 1 by circles. The red solid line in Fig. 1 shows

$$T = \left(\frac{\ell}{L}\right)^2 \frac{2 + 4\frac{z_c}{\ell}\left[1 + \frac{D(0)}{D_B}\frac{z_0}{\ell}\right]}{1 + 4\frac{z_c}{L}\left[1 + 2\frac{D(0)}{D_B}\frac{z_0}{L}\right]} \quad (1)$$

that we obtained by assuming $D(z)=D(0)/(1+\tilde{z}/z_c)$ with $\tilde{z}=\min(z,L-z)$ as suggested by van Tiggelen *et al.*[4] Here D_B is the diffusion coefficient in the absence of macroscopic interferences (i.e., in the limit of $k\ell \gg 1$). $D(0)/D_B$ was determined directly from the numerical results at a sufficiently large $L=10^3\ell$, whereas z_c was a free fit parameter. We used $z_0=\frac{2}{3}\ell$, corresponding to no internal reflections at the sample boundaries. Deviations of the fit from the numerical results do not exceed 3% in the whole range of considered L's,

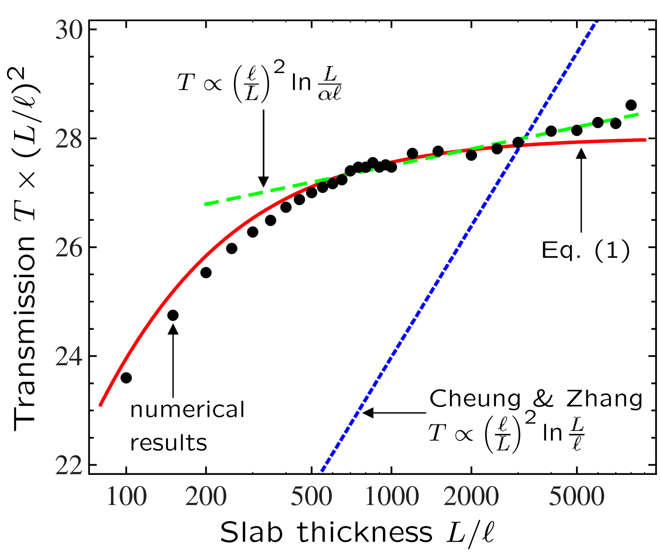

FIG. 1. (Color online) Average transmission coefficient T of a disordered slab of thickness L at the Anderson localization transition. Circles were obtained from the self-consistent theory of localization with a position-dependent diffusion coefficient $D(z)$ by numerical solution (Ref. 6). The solid red line is a fit to the numerical results using Eq. (1) with $D(0)/D_B=0.82$ and $z_c=4.2\ell$. The dotted blue and dashed green lines are fits to numerical data for $L/\ell > 10^3$ using $T\propto(\ell/L)^2\ln(L/\ell)$ and $T\propto(\ell/L)^2\ln(L/\alpha\ell)$, respectively. We obtain $\alpha \simeq 7.37\times10^{-25}$ in the latter case.

which supports the validity of Eq. (1) and its underlying model for $D(z)$. The inaccuracy of the latter model in the middle of the slab cause deviations at small $L<10^3\ell$, whereas deviations at large $L>4\times10^3\ell$ are mostly due to the extremely slow convergence of our computational algorithm for thick slabs and would, most likely, disappear if more computer time were available. We note that $T\times(L/\ell)^2$ grows with $\ln(L/\ell)$ for $L<10^3\ell$, but then saturates at a constant level for larger L, suggesting $T\propto(\ell/L)^2$ in the limit of large L.

Neither the ensemble of numerical results of Fig. 1, nor its small- or large-L parts can be fit by $T=\text{const}\times(\ell/L)^2\ln(L/\ell)$ proposed by CZ. This is easy to see from Fig. 1 where we show a fit of the above equation to our numerical data for $L/\ell>10^3$ (dotted blue straight line). It is clear that the fast growth of $T\times(L/\ell)^2$ with $\ln(L/\ell)$ predicted by CZ is not supported by our numerical calculations: the numerical results only show an increase of 20% in the range of $L/\ell=100-8000$ and 4% in the range $L/\ell=1000-8000$, whereas the result of CZ increases by 100 and 30%, respectively. For large $L>10^3\ell$, a reasonable fit can be achieved by $T\propto(\ell/L)^2\ln(L/\alpha\ell)$. The result of CZ would correspond to $\alpha\sim1$, whereas a fit to the numerical data yields $\alpha\sim10^{-24}\ll1$. This value is unphysically small and implies existence of length scales that are 24 orders of magnitude shorter than the mean-free-path ℓ. We therefore conclude that our numerical results exclude the possibility of logarithmic scaling of $T\times(L/\ell)^2$ with L/ℓ. Appearance of this scaling in Ref. 1 should then be an artifact of replacing $D(\mathbf{r})$ by its harmonic mean.

In conclusion, we have shown the importance of properly treating the position dependence of the diffusion coefficient $D(\mathbf{r})$ in the SC theory of localization. In particular, replacing $D(\mathbf{r})$ by its harmonic mean leads to an incorrect scaling law for the transmission coefficient T with the thickness L of disordered slab at the mobility edge. The correct scaling law $T\propto1/L^2$ is obtained by solving SC equations with a position dependent $D(\mathbf{r})$.

The computations presented in this paper were performed on the cluster HealthPhy (CIMENT, Grenoble). S.E.S. acknowledges financial support from the French ANR (project No. 06-BLAN-0096 CAROL) and the French Ministry of Education and Research.

[1] S. K. Cheung and Z. Q. Zhang, Phys. Rev. B **72**, 235102 (2005).
[2] D. Vollhardt and P. Wölfle, in *Electronic Phase Transitions* (Elsevier Science, Amsterdam, 1992), p. 1.
[3] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[4] B. A. van Tiggelen, A. Lagendijk, and D. S. Wiersma, Phys. Rev. Lett. **84**, 4333 (2000).
[5] S. E. Skipetrov and B. A. van Tiggelen, Phys. Rev. Lett. **92**, 113901 (2004).
[6] S. E. Skipetrov and B. A. van Tiggelen, Phys. Rev. Lett. **96**, 043902 (2006).
[7] N. Cherroret and S. E. Skipetrov, Phys. Rev. E **77**, 046608 (2008).
[8] C. Tian, Phys. Rev. B **77**, 064205 (2008).
[9] S. K. Cheung, X. Zhang, Z. Q. Zhang, A. A. Chabanov, and A. Z. Genack, Phys. Rev. Lett. **92**, 173902 (2004).
[10] In contrast to Ref. 1, we use a two-dimensional (2D) cutoff $q_{\max}=\mu/\ell$ in the integration over momentum $\mathbf{q}$ in Eq. (2) of Ref. 6 and adjust $\mu=1/3$ to obtain the mobility edge in the infinite medium at $k\ell=1$. The need for a cutoff arises from the failure of the small-q approximation implied by Eq. (1) of Ref. 6 for $q>1/\ell$ (small distances). The exact way of applying the cutoff (2D or three-dimensional cutoff, exact value of μ, etc.) does not affect scaling with (large) L as far as the cutoff is consistent with the definition of the mobility edge.